A companion to vectors.

Lesson One.

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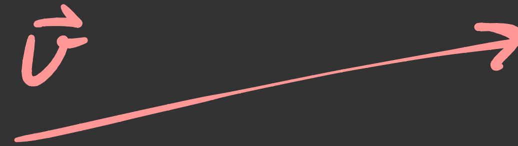


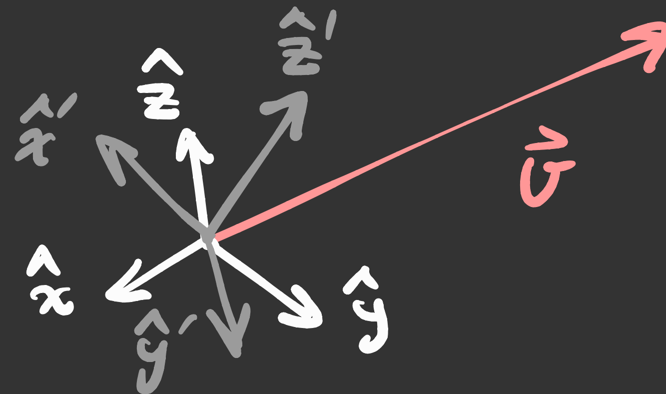
The vector $\vec{v}$ is a geometric object. It has length / magnitude, an orientation / angle relative to other vectors, as well as a dot product with other vectors — but we don't have any of these things until we have defined a metric g! What do we have?

$\vec{v}$ belongs to a vector space V (with all the usual vector space axioms)

If we choose a set of basis vectors, such as $(\hat{x}, \hat{y}, \hat{z})$, we can decompose $\vec{v}$ into

$$\vec{v} = v^x \hat{x} + v^y \hat{y} + v^z \hat{z}.$$

The same vector $\vec{v}$ can be expressed in many different choices of basis.



$$\vec{v} = v^x \hat{x} + v^y \hat{y} + v^z \hat{z}.$$

$$\vec{v} = v^{x'} \hat{x}' + v^{y'} \hat{y}' + v^{z'} \hat{z}'.$$

The sum $\vec{v} = v^x \hat{x} + v^y \hat{y} + v^z \hat{z}$

can be written in a more compact form.

$$\vec{v} = v^x \hat{x} + v^y \hat{y} + v^z \hat{z}$$

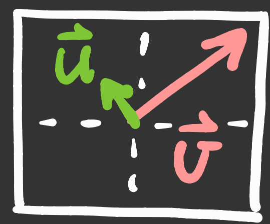
$$\vec{v} = \sum_i v^i \hat{x}_i$$

$$\vec{v} = v^i \hat{x}_i = v^{i'} \hat{x}_{i'} .$$

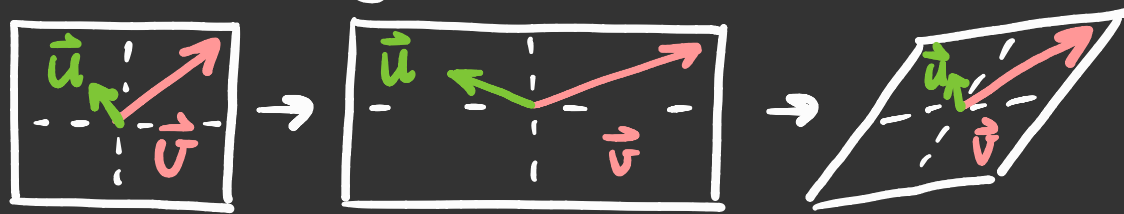
This is the Einstein summation convention:

repeated indices are summed over.

Let's consider the dot product e.g. $\vec{u} \cdot \vec{v}$ as depicted in the diagram. We are used to the formula $\vec{u} \cdot \vec{v} = u^x v^x + u^y v^y + u^z v^z$, but this is only in a specific coordinate system with a specific metric! What does $\vec{u} \cdot \vec{v}$ seem to be? What if we distort the diagram? How do arrive at the right formula?



$\vec{u}$ and $\vec{v}$ may appear to be perpendicular, indicating a dot product of zero, but note what happens if we allow the diagram to distort:



Choosing to depict $\vec{u}$ as $\tilde{u}$ though, we see something else.

Do we see $\vec{u} \cdot \vec{v} = 0$ represented here? What is $\tilde{u}$?

$\tilde{u}$ is the dual vector to $\vec{u}$, in that if we say $\vec{u}$ lives in a vector space V, then $\tilde{u}$ lives in the dual space V^*.

Dual vectors, also called 1-forms, take in vectors as input and return scalars.

$$\tilde{u} : V \to \mathbb{R}$$

It is our companion to vectors.

How do we express $\tilde{u}$, given that we can express $\vec{v}$ as $\vec{v} = v^x \hat{x} + v^y \hat{y} + v^z \hat{z}$?

Just as a vector space V can be spanned by a choice of basis vectors, the dual space V^* is spanned by dual basis vectors.

$$\Rightarrow \quad \tilde{u} = u_x \, dx + u_y \, dy + u_z \, dz$$

How are these defined?.

The dual basis vectors dx^i are defined such that $dx^i(\hat{x}_j) = \delta^i{}_j$, the Kronecker delta symbol: $\boxed{\delta^i{}_j = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}}$.

In this way, we can see how dx^i picks out the i-th component of a vector $\vec{v}$:

$$dx(\vec{v}) = v^x dx(\hat{x}) + v^y \cancel{dx(\hat{y})} + v^z \cancel{dx(\hat{z})}$$

$$dx(\vec{v}) = v^x.$$

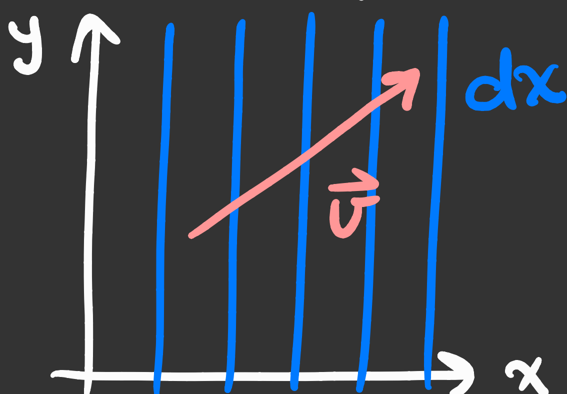
The dual $\tilde{u}$ to a vector $\vec{u}$ is obtained via the metric g: $\boxed{u_i = g_{ij} u^j}$. The metric is expressed as a tensor product of 1-forms:

$g = g_{ij}\, dx^i \otimes dx^j$, then, we have:

$$\begin{aligned} g\vec{u} &= g_{ij}\, dx^i \otimes dx^j\, u^k \hat{x}_k \\ &= g_{ij} u^k dx^i(\hat{x}_k) \otimes dx^j \\ &= g_{ij} u^k \delta^i{}_k dx^j \\ &= g_{ij} u^i dx^j \\ &= u_j dx^j = \tilde{u}. \checkmark \end{aligned}$$

How do we depict the basis 1-forms?

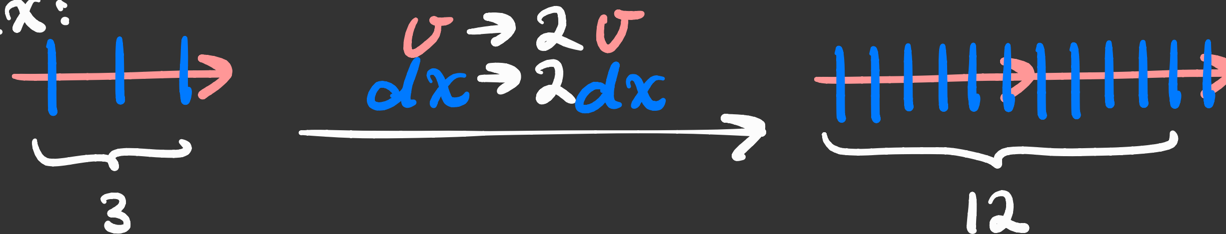
Recall that $dx^i(\vec{v}) = v^i$ — the 1-form projects the vector along a direction. We can then depict, for example dx, as:



$dx(\vec{v}) = 3$, the number of intersections.

The depiction of dx has a length^{-1} nature, cancelling out the dimensions of length a vector $\vec{v}$ has, yielding a dimensionless number.

Ex:



Note that the depiction is a discretized one — while $dx(\vec{v}) \in \mathbb{R}$, the number of intersections $\in \mathbb{Z}$.

Also note that the lines representing dx should be thought of as extending in all directions!

Let's now consider how $\frac{\partial}{\partial x^i}$ defines a vector at each point, even in curvilinear coordinates, for fixed i.

Every vector $\vec{v}$ at a point p corresponds to a possible directional derivative of a function f:

$$\vec{v} \to D_{\vec{v}} f = v^i \frac{\partial}{\partial x^i} f.$$

We then write

$$D_{\vec{v}} f \text{ as } \vec{v} f \Rightarrow \vec{v} = v^i \frac{\partial}{\partial x^i}.$$

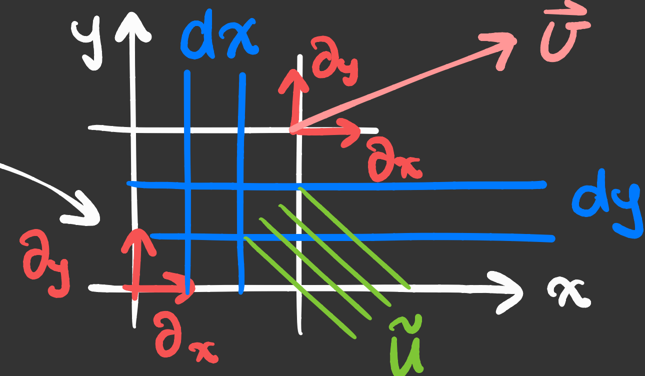
Dual basis vectors are defined such that $\boxed{dx^i(\partial_j) = \delta^i_j}$.

(Note ∂_i is shorthand for $\partial/\partial x^i$)

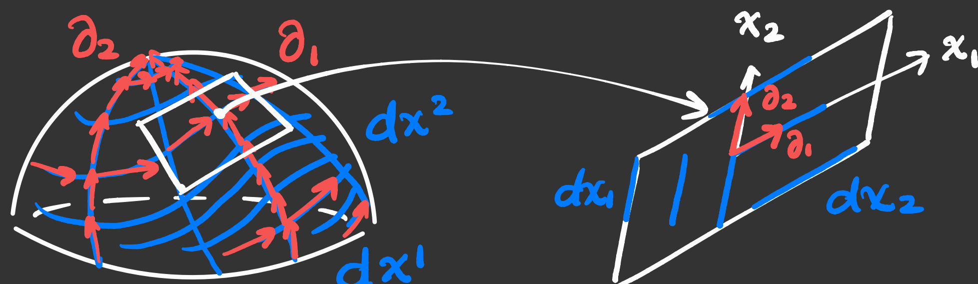
In flat space with a Cartesian coordinate system we have:

Note now

$dx^i \partial_j = \delta^i_j$



Smooth manifolds in space are locally flat.



Note how 1-forms follow lines of constant value.

We will see later that gradients are 1-forms.

We are now ready to discuss the dot product (scalar product) in a generic coordinate system with generic metric.

$$\vec{u} \cdot \vec{v} = u^i \partial_i \cdot v^j \partial_j$$
$$= u^i v^j \partial_i \cdot \partial_j$$
$$= u^i v^j g_{ij}$$
$$= u_j v^j$$

$$\tilde{u}(\vec{v}) = u_i dx^i v^j \partial_j$$
$$= u_i v^j dx^i \partial_j$$
$$= u_i v^j \delta^i_j$$
$$= u_i v^i \quad \checkmark$$

Note:
$$\boxed{g_{ij} \equiv \partial_i \cdot \partial_j}$$

We will see mathematically how $\tilde{u}(\vec{v})$ is preserved under transformations.

Given an arbitrary, differentiable coordinate transformation $x^{\mu'}(x^{\mu})$, the jacobian $\frac{\partial x^{\mu'}}{\partial x^{\mu}}$ is the linear approximation to the transformation at a point.

Tensors, defined at a point, transform with factors of the (inverse) jacobian.

$$x^{\mu'} = x^{\mu'}(x^{\mu})$$

$$v^{\mu'} = v^{\mu} \frac{\partial x^{\mu'}}{\partial x^{\mu}}$$

$$u_{\mu'} = u_{\mu} \frac{\partial x^{\mu}}{\partial x^{\mu'}}$$

We can deduce these rules by examining the chain rule:

$$dx = \frac{\partial x}{\partial x'} dx' + \frac{\partial x}{\partial y'} dy' \Rightarrow dx = \frac{\partial x}{\partial x^{\mu'}} dx^{\mu'}$$

$$\Rightarrow \boxed{dx^{\mu} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} dx^{\mu'}}$$

also, $\boxed{\frac{\partial}{\partial x^{\mu}} = \frac{\partial x^{\mu'}}{\partial x^{\mu}} \frac{\partial}{\partial x^{\mu'}}}$

Then, we have:

$$v^{\mu} \frac{\partial}{\partial x^{\mu}} = v^{\mu} \frac{\partial x^{\mu'}}{\partial x^{\mu}} \frac{\partial}{\partial x^{\mu'}} = v^{\mu'} \frac{\partial}{\partial x^{\mu'}} \Rightarrow \boxed{v^{\mu'} = v^{\mu} \frac{\partial x^{\mu'}}{\partial x^{\mu}}}$$

$$u_{\mu} dx^{\mu} = u_{\mu} \frac{\partial x^{\mu}}{\partial x^{\mu'}} dx^{\mu'} = u_{\mu'} dx^{\mu'} \Rightarrow \boxed{u_{\mu'} = u_{\mu} \frac{\partial x^{\mu}}{\partial x^{\mu'}}}.$$

Let's now reexamine how $\tilde{u}(\vec{v})$ behaves under a coordinate transformation:

$$\tilde{u}'(\vec{v}') = u_{\mu'} v^{\mu'} = u_\mu \frac{\partial x^\mu}{\partial x^{\mu'}} v^\nu \frac{\partial x^{\mu'}}{\partial x^\nu}$$

$$= u_\mu v^\nu \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\mu'}}{\partial x^\nu} = u_\mu v^\nu \delta^\mu{}_\nu$$

$$= u_\mu v^\mu$$

$$= \tilde{u}(\vec{v}) \checkmark$$

$\Rightarrow$ The scalar product is preserved under coordinate transformations!

Finally now we can answer one of the most important questions about vectors: What is its length / magnitude?

$$|\vec{v}| = \sqrt{\vec{v} \cdot \vec{v}} = \sqrt{v_\mu v^\mu} = \sqrt{g_{\mu\nu} v^\mu v^\nu}$$

Note: our metric also transforms under coordinate transformations! As it has two lower indices, $\boxed{g_{\mu'\nu'} = g_{\mu\nu} \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}}}$.

Summary I: vectors and 1-forms (and their depictions!) transform in complementary ways so that scalar products (depicted by number of intersections) are preserved under coordinate transformations!

Summary II: The metric (or inner product), often denoted by g, is the additional structure that is needed to be added to V to define the lengths/magnitudes of vectors, as well as the angle between two vectors. It is the metric that defines the geometry of the space.

$$\vec{v} \xrightarrow{g} \tilde{v}$$

END