A sincerity to interiors.

Lesson Three.

10/24/2022

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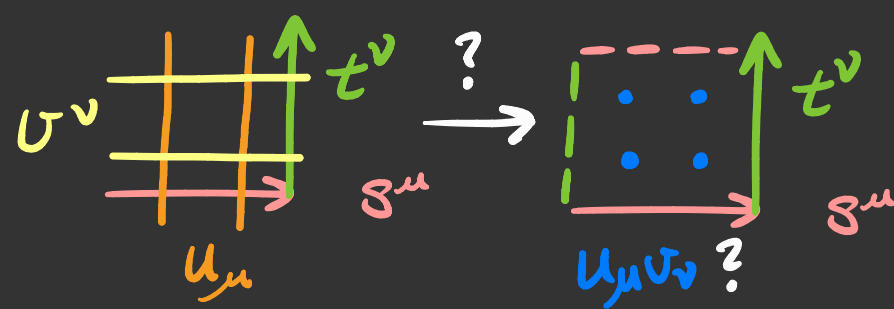
In the last two lessons, we've gotten to know vectors, one-forms, and scalar products quite well. Let us now talk about products of these objects. Consider the product $s^\mu t^\nu u_\mu v_\nu$, shown below.



Both $s^\mu u_\mu$ and $t^\nu v_\nu$ give 2, yielding a product of 4. Could $u_\mu v_\nu$ be an area element?

It cannot be, since $s^\nu t^\mu u_\mu v_\nu = 0$.

Consider the quantity $A_{\mu\nu} := u_\mu v_\nu - v_\mu u_\nu$.

In the case where $\tilde{u} = dx$, $\tilde{v} = dy$, and $\vec{s} = s^\mu \partial_\mu$, $\vec{t} = t^\mu \partial_\mu$, we have

$$A_{\mu\nu} s^\mu t^\nu = u_\mu v_\nu s^\mu t^\nu - v_\mu u_\nu s^\mu t^\nu$$
$$= s^x t^y - s^y t^x,$$

which is the signed area spanned by the two vectors in the x-y plane! We give this antisymmetric product between $\tilde{u}$ and $\tilde{v}$ the name <u>wedge product</u>, and it is written as $\tilde{u} \wedge \tilde{v}$.

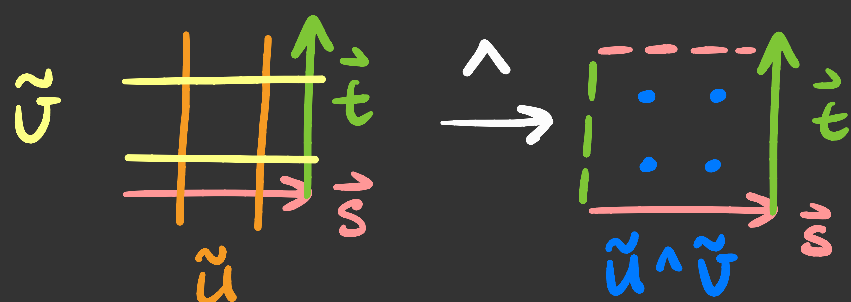
In tensor product notation, we have:

$$A = \tilde{u} \wedge \tilde{v} \,, \quad \text{and} \quad A = \tfrac{1}{2} A_{\mu\nu}\, dx^{\mu} \wedge dx^{\nu}.$$

$$= \tilde{u} \otimes \tilde{v} - \tilde{v} \otimes \tilde{u} \Rightarrow A_{\mu\nu} = u_{\mu} v_{\nu} - v_{\mu} u_{\nu}.$$

Notice that the definition of the wedge product means for any one-forms $\tilde{u}$ and $\tilde{v}$, $\tilde{u} \wedge \tilde{u} = 0$, and $\tilde{u} \wedge \tilde{v} = -\tilde{v} \wedge \tilde{u}$. In our example, $A = dx \wedge dy$ was the area element in the x-y plane. Just as dx, dy, etc are our basis one-forms, elements such as $dx \wedge dy$, $dy \wedge dz$, etc are our basis <u>two-forms</u>.

Just as we depicted scalar products between vectors and one-form as a number of intersections, we will depict scalar products (e.g. $A_{\mu\nu} s^\mu t^\nu$) between vectors and two-forms as a number of intersections between the two-form and the signed area spanned by the two vectors.



We can also obtain a two-form from a one-form by using an operator called the exterior derivative.

On one-forms, it is given by:

$$\boxed{d(\omega_\nu dx^\nu) = \partial_\mu \omega_\nu dx^\mu \wedge dx^\nu}.$$

Let us now compute $d(d\phi) = d(\partial_\nu \phi dx^\nu)$.

$$\begin{aligned} d(\partial_\nu \phi dx^\nu) &= \partial_\mu \partial_\nu \phi dx^\mu \wedge dx^\nu \\ &= -\partial_\nu \partial_\mu \phi dx^\nu \wedge dx^\mu \\ &= -\partial_\mu \partial_\nu \phi dx^\mu \wedge dx^\nu \end{aligned}$$

$\Rightarrow d(d\phi)$ vanishes!

As an explicit example, let's compute $F = dA$ for the case where A exists in the 2D plane, that is:

$$A = A_x dx + A_y dy.$$

$$dA = \partial_\mu A_\nu dx^\mu \wedge dx^\nu,$$ and μ and ν range over (x, y)

$$dA = \partial_\mu A_x dx^\mu \wedge dx + \partial_\mu A_y dx^\mu dy$$

$$dA = \partial_x A_x dx \wedge dx + \partial_x A_y dx \wedge dy$$
$$+ \partial_y A_x dy \wedge dx + \partial_y A_y dy \wedge dy.$$

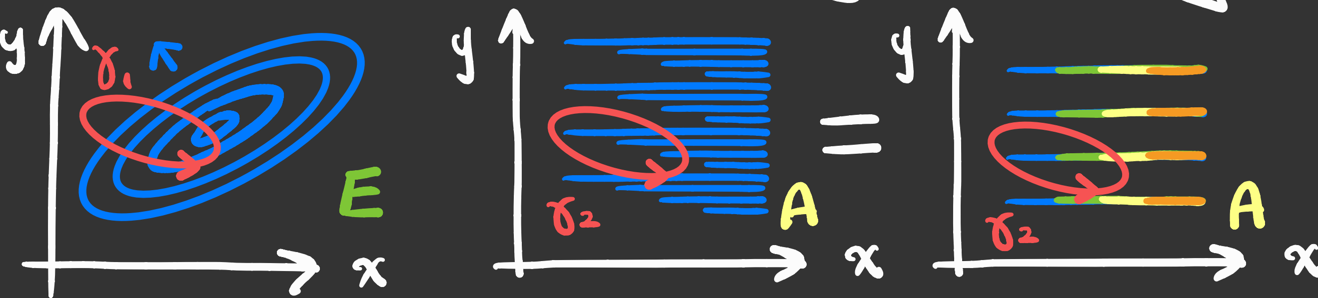
Since $dx \wedge dx$ and $dy \wedge dy = 0$, and $dx \wedge dy = -dy \wedge dx$, we finally have: $F = (\partial_x A_y - \partial_y A_x)\, dx \wedge dy$. The exterior derivative of a one-form computes the curl!

In our last lesson we encountered the differential forms $d\phi = \partial_\mu \phi \, dx^\mu$, $A = A_\mu dx^\mu$, and F, which we now know how to express in terms of basis 2-forms. We will get to know these differential forms even better by integrating them. That is, by considering the integrals

$$\int d\phi \,, \quad \int A \,, \quad \text{and} \quad \int F \,.$$

This will also illuminate the nature of the exterior derivative, and we will see how to depict it.

Let us consider the following three diagrams.



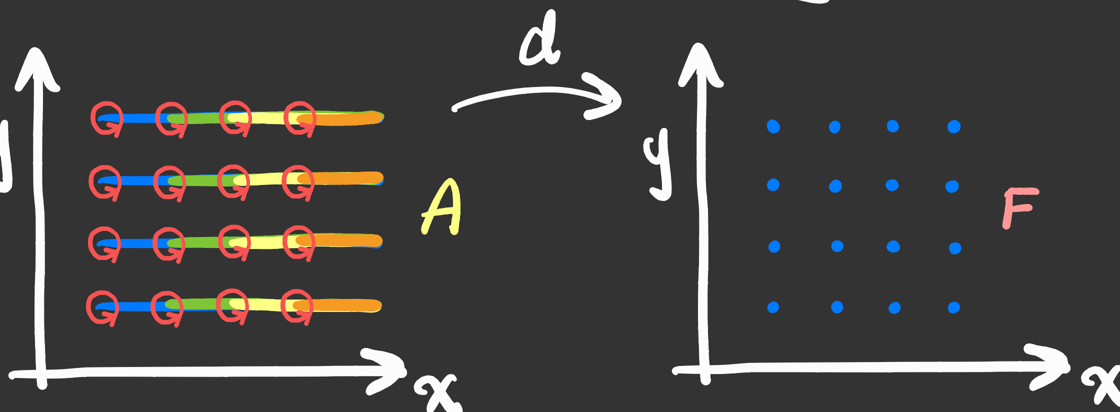
Just as we depicted scalar products between 1-forms and vectors as the number of intersections between them, Integrals of differential forms over some manifold can be depicted as the number of intersections between them.

Let us consider $\oint_{\gamma_1} E = \oint_{\gamma_1} E_i dx^i$ and $\oint_{\gamma_2} A = \oint_{\gamma_2} A_i dx^i$. E has the property that any closed loop (e.g. γ_1) crosses its curves an even number of times, once in each direction. As each positive contribution to the loop integral has a corresponding negative contribution, all such integrals turn out to be zero! Meanwhile A lacks this feature. A has non-zero circulation! Let us see how we can depict the associated curl.

Recall Stokes' Theorem for the curl operator: $$\iint_M (\vec{\nabla} \times \vec{A}) \cdot d\vec{a} = \oint_{\partial M} \vec{A} \cdot d\vec{l}$$

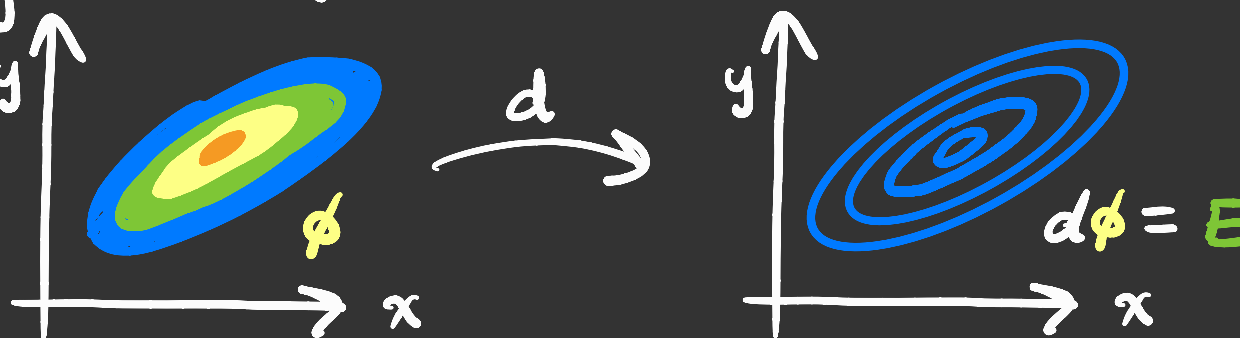
If our closed loop integral has a non-zero circulation, it means that the area it encloses contains curl. These loops can be made arbitrarily small as long as they enclose precisely the feature that the gradient of a field doesn't have — the endpoints of the curves that denote that these curves don't close!

Identifying these features in our diagram of A, we can now understand the diagram for $dA = F$.



We conclude then, that endpoints in our diagrams of differential 1-forms are indicative of curl at that location, and the exterior derivative picks out precisely these points.

Returning to our diagram for E, we note that as E has zero curl, it can be represented as the gradient of some scalar field!



We can then recognize $dE = dd\phi = 0$ is the differential forms version of the identity $\vec{\nabla} \times \vec{\nabla} \phi = 0$.
We will also see that $dF = 0$, giving $ddA = 0$ too!

Let us compute dF.

$$dF = \partial_\alpha \left(\tfrac{1}{2}F_{\mu\nu}\right) dx^\alpha \wedge dx^\mu \wedge dx^\nu$$
$$= \tfrac{1}{2}\partial_\alpha \left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) dx^\alpha \wedge dx^\mu \wedge dx^\nu$$
$$= \tfrac{1}{2}\partial_\alpha \partial_\mu A_\nu \, dx^\alpha \wedge dx^\mu \wedge dx^\nu$$
$$- \tfrac{1}{2}\partial_\alpha \partial_\nu A_\mu \, dx^\alpha \wedge dx^\mu \wedge dx^\nu$$

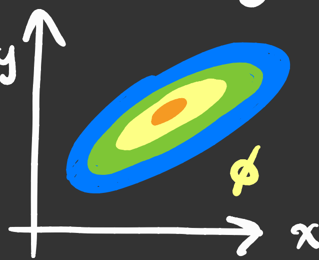
Which becomes zero for the same reason we saw $\partial_\mu \partial_\nu \phi \, dx^\mu \wedge dx^\nu$ earlier in this lesson: partial derivatives commute, while the wedge product is antisymmetric! As this will occur for any differential form, we say $\boxed{d^2 = 0}$.

Let us summarize the relations we've explored so far:

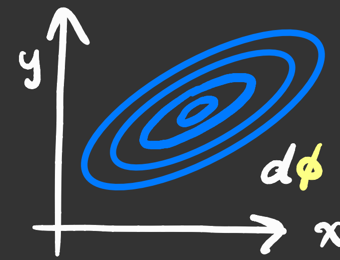
0-form

1-form

$\xrightarrow{\cancel{d}}$



$\xrightarrow[\vec{\nabla}]{d}$



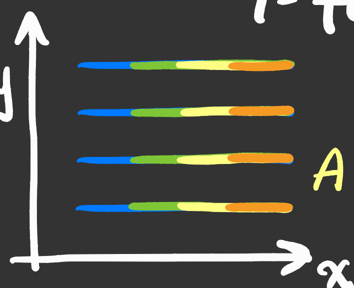
$\xrightarrow[\vec{\nabla}\times]{d}$ 0

$\vec{\nabla}\times\vec{\nabla}\phi=0$

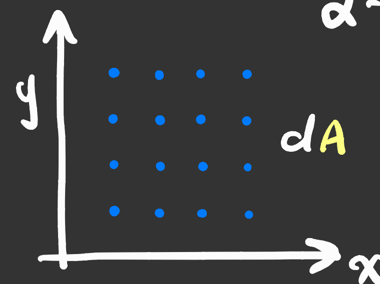
1-form

2-form

$\xrightarrow{\cancel{d}}$



$\xrightarrow[\vec{\nabla}\times]{d}$



$\xrightarrow[\vec{\nabla}\cdot]{d}$ 0

$\vec{\nabla}\cdot\vec{\nabla}\times\vec{A}=0$

What about 3-forms?

Let us recall Stokes' theorem in its familiar vector calculus forms first.

$$\int_{\gamma} \vec{\nabla}\phi \cdot d\vec{l} = \phi(B) - \phi(A)$$

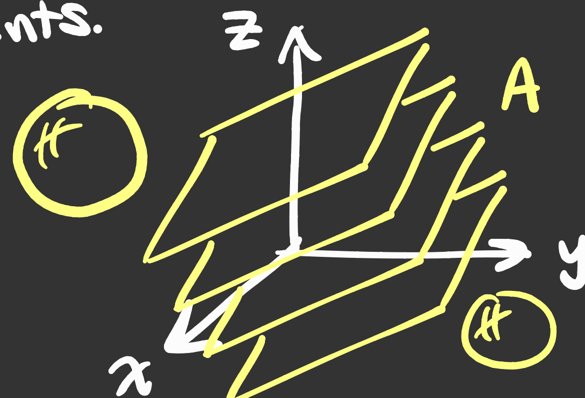
$$\iint_{M} (\vec{\nabla} \times \vec{B}) \cdot d\vec{a} = \oint_{\partial M} \vec{B} \cdot d\vec{l}$$

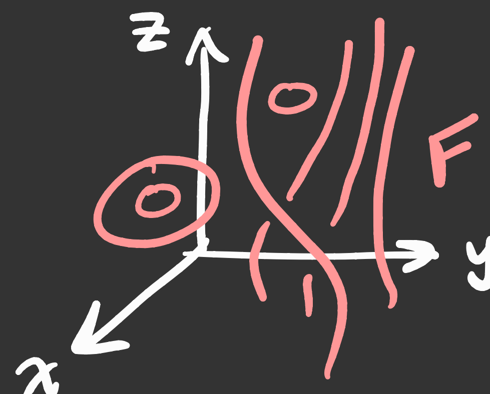
$$\iiint_{M} \vec{\nabla} \cdot \vec{E} \, dV = \oiint_{\partial M} \vec{E} \cdot d\vec{a}$$

Three forms (e.g. $\rho = dx \wedge dy \wedge dz$) are volume forms and appear when 2-forms have a non-zero divergence.

Thinking of basis 1-forms as surfaces of constant coordinates, we know that 1-forms are depicted as surfaces in 3D as opposed to lines in 2D.

Thinking of basis 2-forms as wedges of basis 1-forms, we also know 2-forms are depicted as lines as opposed to single points.

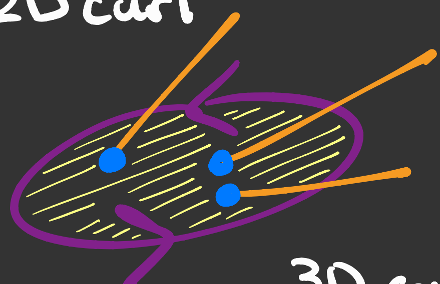




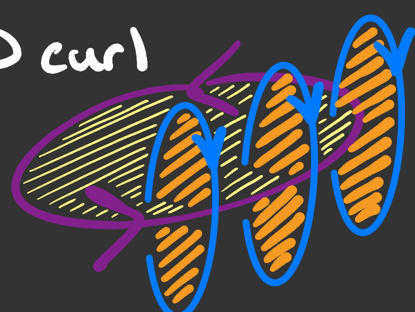
Depicting two-forms in this way in 3D, as one-dimensional curves intersected by / integrated with two-dimensional surfaces, allows us to understand Stokes' theorem in its differential form:

$$\boxed{\int_M d\omega = \int_{\partial M} \omega}$$

2D curl



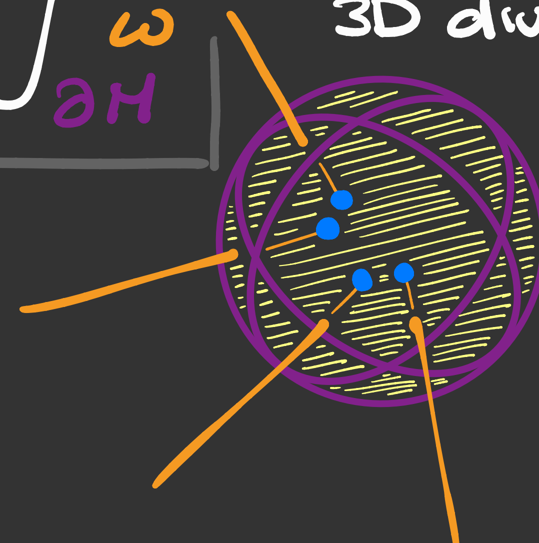
3D curl



1-form

2-form

3D div



2-form

3-form

Summary: Developing the language of differential forms led us to the exterior derivative, which we found identifies the "exterior" (endpoints, boundaries) of our depiction of a p-form, and tells us that these exteriors depict our p+1 form! Stokes' Theorem can be cast into this language, which tells us when the exterior (derivative) of a form lies in the interior of an integration manifold, an integral of the form on the manifold's boundary will give the same answer!

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