A timeliness to potentials.

Lesson Two.

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Gabe Bonilla

In our last lesson we encountered the metric $g = g_{ij}\, dx^i \otimes dx^j : V \to V^*$, which took vectors to their duals, 1-forms. Now we will consider a specific metric η, The Minkowski metric.

$\eta = \eta_{\mu\nu}\, dx^\mu \otimes dx^\nu$ carries Greek indices denoting (t, x, y, z) and describes a spacetime geometry!

η is defined by the mapping:

$$\boxed{\begin{array}{ll} \partial_t \xrightarrow{\eta} -dt, & \partial_x \xrightarrow{\eta} dx, \\ \partial_y \xrightarrow{\eta} dy, & \partial_z \xrightarrow{\eta} dz. \end{array}}$$

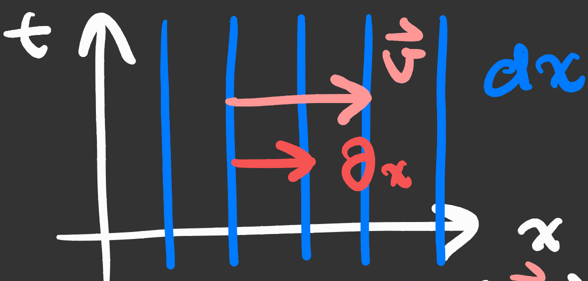
Let's explore the metric η by computing the magnitudes of a couple of vectors.

if $\vec{v} = 2\partial_x$:

$\Rightarrow |\vec{v}|^2 = \tilde{v}(\vec{v}) = \eta\,\vec{v}\,\vec{v}$

and $\tilde{v} = 2\,dx$

$\Rightarrow |\vec{v}|^2 = 2\,dx(2\partial_x) = 4$



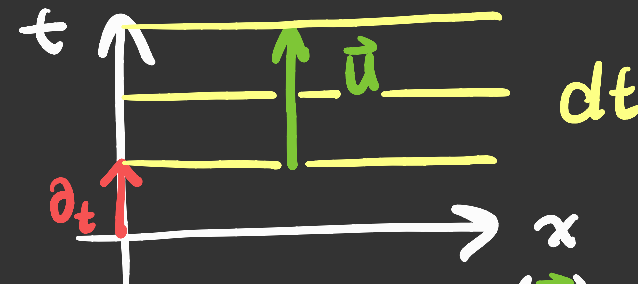
We can see here $dx(\vec{v}) = 2$.

if $\vec{u} = 2\partial_t$:

$\Rightarrow |\vec{u}|^2 = \tilde{u}(\vec{u}) = \eta\,\vec{u}\,\vec{u}$

and $\tilde{u} = -2\,dt$

$\Rightarrow |\vec{u}|^2 = -2\,dt(2\partial_t) = -4.$



We can see here $-dt(\vec{u}) = -2$.

The vector $\vec{u}$ has a negative squared norm!
$\vec{u}$ is an example of a timelike vector.
As $\tilde{u}(\vec{u})$ is a scalar, we know $\vec{u}$ is a timelike vector in all coordinate systems. Let us now compute the norm of a vector $\vec{\ell} = \partial_t + \partial_x$.

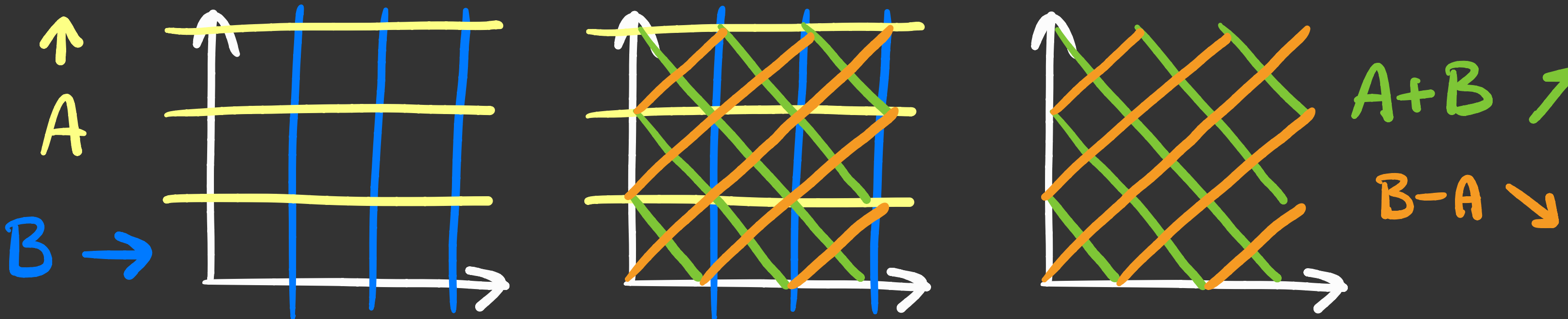
$$|\vec{\ell}|^2 = \tilde{\ell}(\vec{\ell}) = \eta \vec{\ell}\vec{\ell} = \eta(\partial_t + \partial_x)(\partial_t + \partial_x)$$

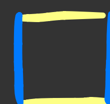
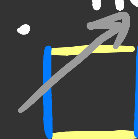
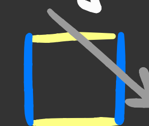
$$= (-dt + dx)(\partial_t + \partial_x)$$

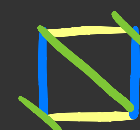
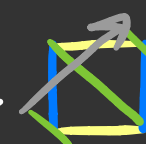
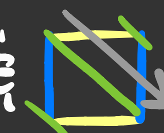
$$= -dt\,\partial_t - \cancel{dt\,\partial_x} + \cancel{dx\,\partial_t} + dx\,\partial_x = 0.$$

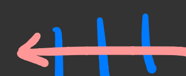
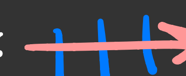
$\Rightarrow \vec{\ell}$ is an example of a null vector! Let's see how 1-form addition can help us understand null vectors.

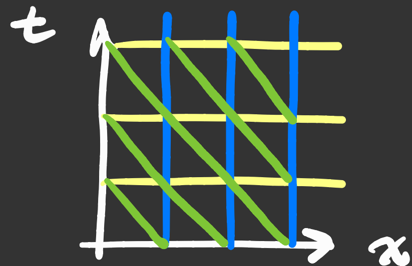
Here we describe how to add 1-forms together.

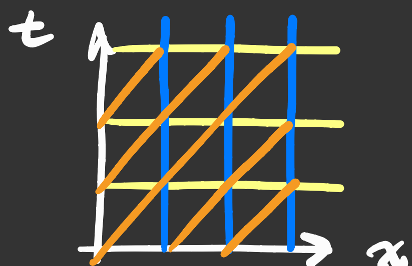


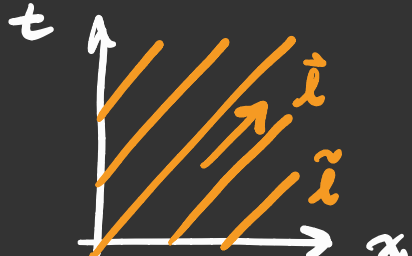
We consider the unit: . There are two ways to intersect both A and B,  = +2 and  = 0.

The unit that fulfills both is: . See  & .

Note that this is with A and B having the orientations $\uparrow$ and $\rightarrow$ as given. 1-forms have a sense of direction just as vectors do:  = -1 × .







dt, dx, and dx+dt are depicted here. We can also draw −dt+dx, by connecting the diagonals across the units in the other direction. We can see now how $\tilde{\ell}(\vec{\ell}) = 0$, they don't intersect at all!

We summarize:

$$\boxed{\begin{aligned} |\vec{v}|^2 > 0 &\Rightarrow \text{spacelike} \\ |\vec{u}|^2 < 0 &\Rightarrow \text{timelike} \\ |\vec{\ell}|^2 = 0 &\Rightarrow \text{lightlike} \end{aligned}}$$

Two coordinate systems with metric η, one moving at velocity $\vec{v} = v\hat{x}$ relative to the other, are related by a Lorentz transformation:

$$\Lambda: \quad \begin{matrix} t' = \gamma t - \gamma\beta x & y' = y \\ x' = \gamma x - \gamma\beta t & z' = z \end{matrix}, \quad \Lambda^{\mu'}{}_{\mu} = \begin{pmatrix} \gamma & -\gamma\beta & & \\ -\gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

where $\beta = v/c$, c = speed of light, and we have $\gamma = (1-\beta^2)^{-1/2}$ chosen units where $c=1$.

(such a Λ is sometimes called a hyperbolic rotation.)

Let us consider the vector $p^\mu = m\partial_t$, with m being a mass term. What is $p^{\mu'}$ in the moving frame?

$$p^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^{\mu}} p^{\mu} = \Lambda^{\mu'}{}_{\mu}\, p^{\mu}$$

$$\Rightarrow p^{\mu'}\partial_{\mu'} = \gamma m \partial_t - \gamma\beta m \partial_x .$$

if $v \ll c$, $\gamma = (1 - v^2/c^2)^{-1/2} \approx 1 + \frac{1}{2} v^2/c^2 .$

$$\Rightarrow p^{\mu'}\partial_{\mu'} \approx \left(m + \tfrac{1}{2} m v^2/c^2\right)\partial_t - \left(mv/c + \cancel{\tfrac{1}{2} m v^3/c^3}\right)\partial_x$$

$$\Rightarrow p^{\mu'} c^2 \partial_{\mu'} \approx \underbrace{mc^2}_{\text{rest mass energy}} + \underbrace{\tfrac{1}{2} m v^2}_{\text{The non-relativistic kinetic energy!}} \partial_t - \underbrace{mv}_{\text{The non-relativistic momentum!}} c\, \partial_x$$

We see that a particle's spatial momentum arises from its motion through time being (hyperbolically) rotated into a spatial dimension!

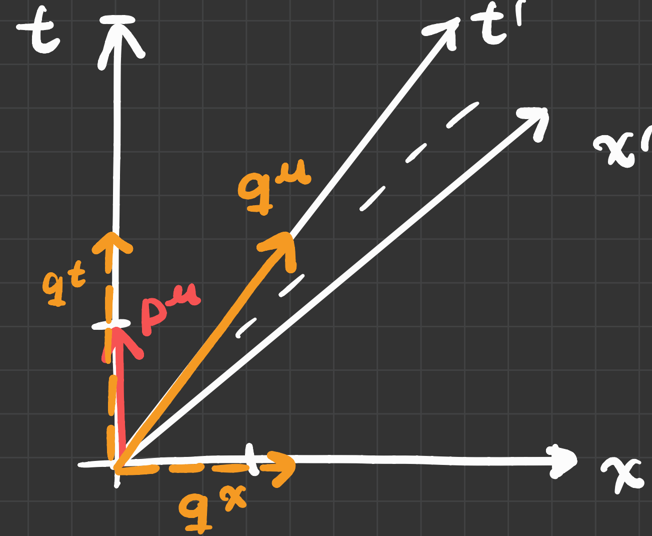
Let's do a relativistic example.

If $v = 4/5\, c$, $\gamma = 5/3 \Rightarrow$

if $q^{\mu'} = m\,\partial_{t'}$, $p^{\mu} = m\,\partial_t$,

$q^{\mu'} = \gamma m\,\partial_t + \gamma\beta m\,\partial_x$.

$q^{\mu'} = 5/3\, m\,\partial_t + 4/3\, m\,\partial_x$.



p^{μ} and $q^{\mu'}$ are examples of 4-momentum vectors, each seen at rest in their own frame, both seen at rest in their own frame, whose time component gives rise to spatial momentum ($p^{x'}$, q^x) when seen in a moving frame. What other physical quantities might behave like this?

Having developed an intuition for how vectors such as $p^\mu \partial_\mu$ and $q^\mu \partial_\mu$ transform, we will also look at how 1-forms such as $A = A_\mu dx^\mu$ transform, and how to plot their components.

Suppose we have $A' = -\phi' dt'$. What is A, the 1-form in the rest frame? ($v = 4/5\, c$, $\gamma = 5/3$).

$$A_\mu = \Lambda^{\mu'}_{\ \mu} A_{\mu'} \Rightarrow A_\mu dx^\mu = -\gamma \phi' dt + \gamma\beta \phi' dx$$

$$\Rightarrow A_\mu dx^\mu = -5/3\, \phi' dt + 4/3\, \phi' dx.$$

What is this object A?

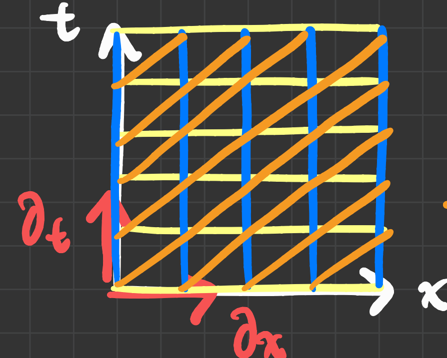
$A_\mu = g_{\mu\nu} A^\nu$ is the dual to $A^\mu = (\phi, \vec{A})$, where ϕ is the scalar potential and $\vec{A}$ is the vector potential! They fulfill the following equations:

$$\boxed{\begin{gathered} \vec{E} = -\vec{\nabla}\phi - \partial_t \vec{A} \\ \vec{B} = \vec{\nabla} \times \vec{A} \end{gathered}}$$

In the relativistic formulation of electrodynamics the vector potential $\vec{A}$ is promoted to the four-potential A^μ by noting that the scalar potential ϕ is actually the time component A^t of the four-potential.

With $A_{\mu'} dx^{\mu'} = -\phi' dt'$ and
$A_\mu dx^\mu = -5/3\,\phi' dt + 4/3\,\phi' dx$,
and additionally assuming that $\phi' = 1$,
we are ready to plot A in both coordinate systems.
With $dx^\mu(\partial_\nu) = \delta^\mu{}_\nu$, we have $-5/3\,dt(\partial_t) = -5/3$ and $-5/3\,dt(3\partial_t) = -5 \Rightarrow 3\partial_t$ traverses -5 units demarcated by $-5/3\,dt$. Likewise, $4/3\,dx(3\partial_t) = 4$.

We can represent these relations with:



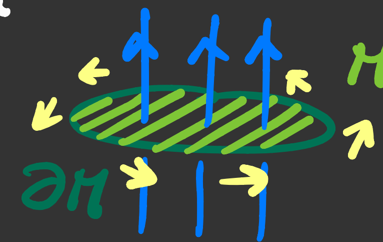
$-5/3\,dt$, $4/3\,dx$

$-5/3\,dt + 4/3\,dx = dt'$.

Why do we want to describe the four-potential A as a 1-form? Consider the equation: $\vec{B} = \vec{\nabla} \times \vec{A}$.

We can integrate both sides over a 2D manifold M.

$$\iint_M \vec{B} \cdot d\vec{a} = \iint_M \vec{\nabla} \times \vec{A} \cdot d\vec{a}$$



We can then apply Stokes' theorem:

$$\iint_M \vec{B} \cdot d\vec{a} = \oint_{\partial M} \vec{A} \cdot d\vec{\ell}$$

We notice that $\vec{A} \cdot d\vec{\ell} = A_i dx^i$, a 1-form! So we will be interested in evaluating integrals of the form $\int_\gamma A_\mu dx^\mu = \int_\gamma A$ for some path γ.

The second equation to consider is the equation:

$$\vec{E} = -\vec{\nabla}\phi - \partial_t \vec{A}.$$

Looking at $\vec{\nabla}\phi$, we find that

$$\vec{\nabla}\phi = \frac{\partial \phi}{\partial x}\hat{x} + \frac{\partial \phi}{\partial y}\hat{y} + \frac{\partial \phi}{\partial z}\hat{z}.$$

The dual to $\vec{\nabla}\phi$ with the flat space ($g=\mathbb{1}$) metric is:

$$\mathbb{1}\vec{\nabla}\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz = \partial_i \phi \, dx^i$$

By the chain rule, this is just the differential $d\phi$, which is a 1-form, depicted as level curves of ϕ!

Finally, let's discuss the term $\partial_t \vec{A}$.

In index notation, this is $\frac{\partial A^i}{\partial x^t}$ (where $x^t \equiv t$).

We also have $\vec{\nabla}\phi$, which is $\frac{\partial A^t}{\partial x^i}$. Then, we have

$$E^i = -\frac{\partial A^t}{\partial x^i} - \frac{\partial A^i}{\partial x^t} = -\partial_i A^t - \partial_t A^i.$$

$\vec{B} = \vec{\nabla} \times \vec{A}$ gives us:

$$B^i = \frac{\partial A^k}{\partial x^j} - \frac{\partial A^j}{\partial x^k} = \partial_j A^k - \partial_k A^j.$$

(Note that in flat space upper and lower indices carry no additional meaning.)

The equations $E^i = -\partial_i A^t - \partial_t A^i$ and $B^i = \partial_j A^k - \partial_k A^j$ are unified in relativistic electrodynamics by the following equation:

$$\boxed{F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu}$$

This is the electromagnetic field strength tensor, the Faraday tensor, and it is a 2-form!

In the next lesson, we will see how to express F in terms of basis 2-forms.

The potential equations ($\vec{E} = -\vec{\nabla}\phi - \partial_t \vec{A}$, $\vec{B} = \vec{\nabla}\times\vec{A}$) have introduced us to three <u>differential forms</u>:

$$d\phi = \partial_\mu \phi \, dx^\mu, \qquad \text{a 1-form}$$

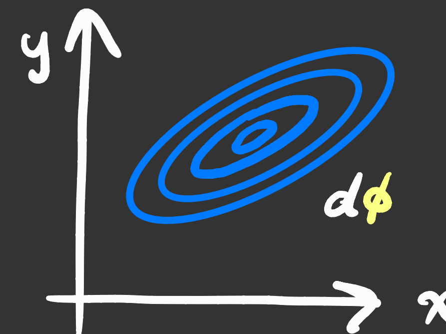
$$A = A_\mu dx^\mu, \qquad \text{a 1-form}$$

$$F, \qquad \text{a 2-form.}$$

They are called so because they can be placed underneath an integral sign and integrated, as we saw with $\int_\gamma A$. How do we visualize these?

Just as 1-forms such as dx were visualized as level curves of x, $d\phi$ can be depicted the same way:

$\xrightarrow{d}$



$A = A_\mu dx^\mu$ we will visualize the same way, with one important difference: the curves depicting A will not necessarily close! We will see this next time.

Summary: The Minkowski metric η highlights how vectors and 1-forms (and their depictions!) are both needed to describe the geometry of spacetime.

We also began to see how electromagnetism could be described using differential forms, something we first explored by considering the timeliness of potentials.

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